Mechanics I

FIZIKA SJPO Training

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Contents

1 Notes

1.1 1D Kinematics

In **kinematics,** we are interested in studying properties of motion, without caring about where the motion comes from. We shall first look at a simple case of 1D kinematics, where motion is constrained to be along a line.

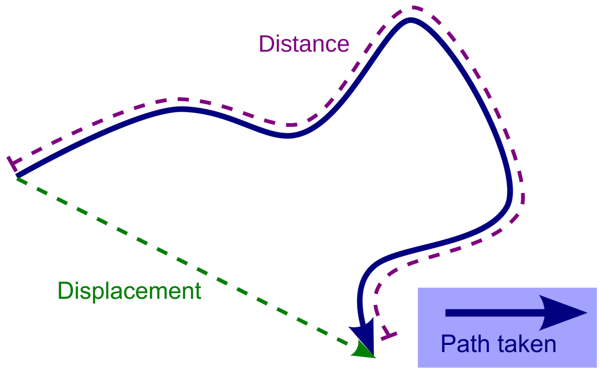
1.1.1 Physical Quantities

The physical quantities (and their corresponding notations) that we shall deal with for kinematics are the following:

1. Distance, s or x
2. Displacement, $\boldsymbol{s}$ or $\boldsymbol{x}$
3. Speed, v or u
4. Velocity, $\boldsymbol{v}$ or $\boldsymbol{u}$
5. Acceleration, $\boldsymbol{a}$
6. Time, t or T

Note which of the above are scalars and which are vectors!

Displacement is defined as the change in an object's position, with its magnitude and direction defined by the shortest line drawn between the initial and final points of the object's position.



As you can see, displacement is not concerned about *how* the object moves, but only where it starts and where it ends. Consequently, you may observe that **the magnitude of the displacement must be always less than or equal to the distance.**

Velocity is defined as the rate of change of an object's position, i.e.

$$\boldsymbol{v} = \frac{\Delta \boldsymbol{x}}{\Delta t} \tag{1}$$

We should also distinguish between the **average** velocity and **instantaneous** velocity.

Average velocity is defined as the total displacement divided by the total time, i.e.

$$\boldsymbol{v}_{\text{ave}} = \frac{\Delta \boldsymbol{x}_{\text{total}}}{\Delta t} \tag{2}$$

Instantaneous velocity is defined as the current rate of change of position, which can be visualised by taking a very small time interval Δt. Formally,

$$\boldsymbol{v}_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{x}}{\Delta t} \tag{3}$$

Likewise, we can write very similar definitions linking acceleration to velocity.

Acceleration is defined as the rate of change of an object's velocity, i.e.

$$\boldsymbol{a} = \frac{\Delta \boldsymbol{v}}{\Delta t} \tag{4}$$

Average acceleration is defined as the total change in velocity divided by the total time, i.e.

$$\boldsymbol{a}_{\text{ave}} = \frac{\Delta \boldsymbol{v}_{\text{total}}}{\Delta t} \tag{5}$$

Instantaneous acceleration is defined as the current rate of change of velocity, i.e.

$$\boldsymbol{a}_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} \tag{6}$$

The mathematics may be a little complicated now, but it will become clearer when we discuss graphical interpretations.

The good thing about the 1D case is that we know the direction of motion – it is constrained along a specified line. As such, vector quantities are not very meaningful here. It suffices for you to think purely in terms of scalars, while being careful that a negative scalar value means that the quantity is opposite to the defined positive direction.

1.1.2 SUVAT Equations

To simplify our analysis, at the SJPO level, we will only consider the case of **constant acceleration** in 1D kinematics. Under constant acceleration, we are able to write the following SUVAT equations:

$$v_{\text{f}} = v_{\text{i}} + at \tag{7}$$

$$x_{\text{f}} = x_{\text{i}} + v_{\text{i}}t + \frac{1}{2}at^2 \tag{8}$$

$${v_{\text{f}}}^2 = {v_{\text{i}}}^2 + 2ax \tag{9}$$

$$x = \frac{1}{2}\left(v_{\text{i}} + v_{\text{f}}\right)t \tag{10}$$

where the subscripts i and f are used to denote initial and final quantities respectively.

It is very important to have Equations (7) to (10) memorised! And, depending on the situation (which variables you have and which variables you need to find), it is important to choose the right equation to use.

Example 1.1. At the instant a traffic light turns green, an automobile starts moving from rest with acceleration $2.2\,\text{m/s}^2$. At the same instant a truck, traveling at constant speed $9.5\,\text{m/s}$ overtakes and passes the automobile. (a) How far beyond the starting point will the automobile overtake the truck? (b) How fast will the automobile be travelling at that instant?

Solution. (a) Let the time elapsed from the instant the traffic light turns green be t. For describing the motion of the automobile with constant acceleration, we shall use Equation (8):

$$x_{\text{auto}} = \frac{1}{2}at^2$$

For describing the motion of the truck with constant speed (and hence 0 acceleration), we shall also use Equation (8):

$$x_{\text{truck}} = v_{\text{i}}t$$

The two vehicles will overtake each other when their displacements (relative to the traffic light) become equal, i.e. $x_{\text{auto}} = x_{\text{truck}}$. Hence, we have:

$$\frac{1}{2}at^2 = v_{\text{i}}t \quad \implies \quad t = 0\,(\text{reject})\ \text{or}\ t = \frac{2v_{\text{i}}}{a}$$

We reject $t = 0$ as this corresponds to the initial point where the truck first passes the automobile. Plugging this into either expression, we get

$$x = \frac{2{v_{\text{i}}}^2}{a} = \frac{2\,(9.5)^2}{2.2} = 82\,\text{m}$$

(b) For describing the speed of the automobile, we shall use Equation (7):

$$v_{\text{auto}} = at$$

To find the speed of the automobile at the instant of overtaking, we substitute in the relevant time t, and hence we get:

$$v_{\text{auto}} = a\left(\frac{2v_i}{a}\right) = 2v_i = 2\,(9.5) = 19\,\text{m/s}$$

Example 1.2. The superhero Green Lantern steps from the top of a tall building. He falls freely from rest to the ground, falling half the total distance to the ground during the last Δt seconds of his fall. What is the height h of the building? Express your answer in terms of g and Δt.

Solution. Let the speed of Green Lantern at a height $\frac{h}{2}$ above the ground (when he has fallen halfway) be v. By considering Equation (9) for the first half of the fall, since he starts from rest, we have:

$$v^2 = 2g\left(\frac{h}{2}\right) = gh$$

The second half of the fall takes a time Δt, and he starts at a speed of v. We can use Equation (8) to describe this:

$$\frac{h}{2} = v\Delta t + \frac{1}{2}g\,(\Delta t)^2$$

Now, it becomes a matter of solving these two equations simultaneously. There are two equations and two unknowns (h and v), which guarantees that we can find a unique solution.

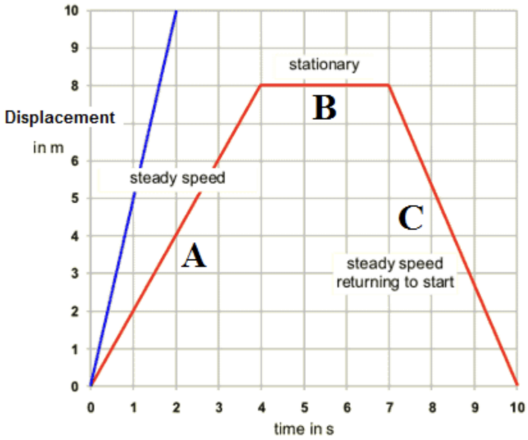
After some tedious algebra (which you should check by yourself), you will get

$$h = \left(3 + 2\sqrt{2}\right) g\,(\Delta t)^2$$

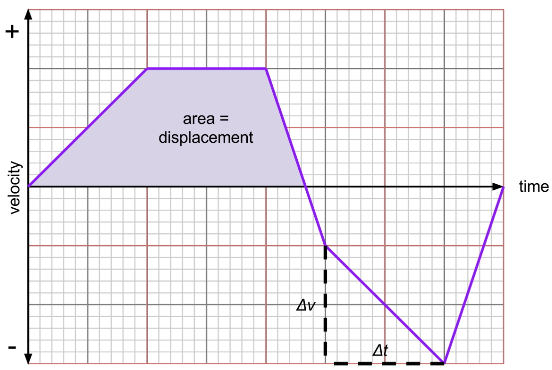
1.1.3 Graphical Interpretations

Often times, you will be given graphs of kinematics quantities plotted against time. It is very important to understand how to interpret such graphs.

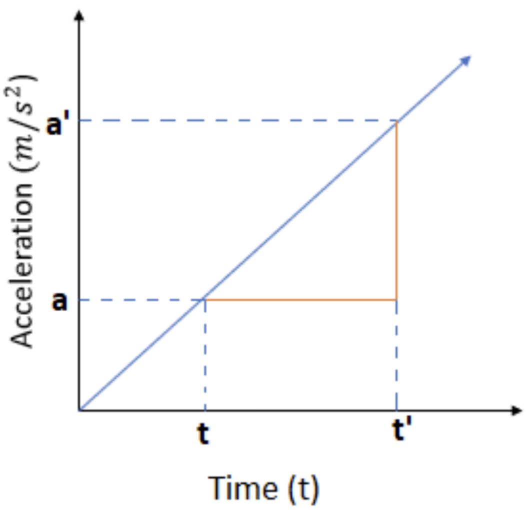
The **gradient** of a **displacement-time graph** gives the **velocity.**



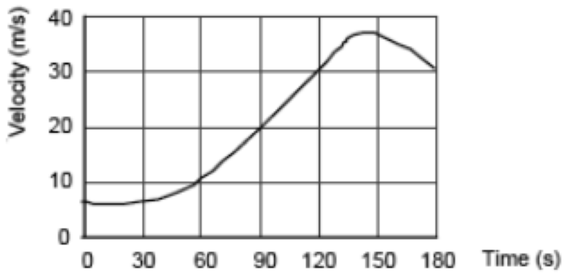
The **area under** a **velocity-time graph** gives the **displacement,** and the **gradient** of a **velocity-time graph** gives the **acceleration.**



The **area under** an **acceleration-time graph** gives the **change in velocity.**



Example 1.3 ($F = ma$ 2007)**.** The graph below shows the velocity-time graph of a car moving in a straight line. What is the (approximate) acceleration of the car at time $t = 90\,\mathrm{s}$?

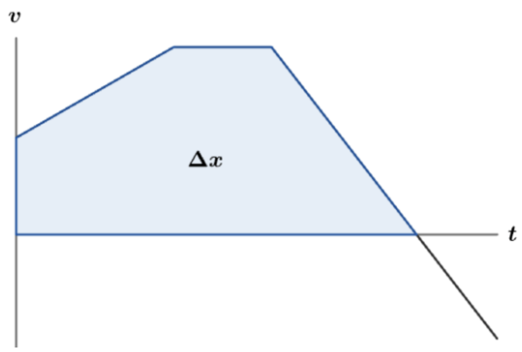


Solution. The acceleration is the gradient of the velocity-time graph. We can approximate the gradient of the graph at $t = 90\,\mathrm{s}$ by drawing a tangent line to the graph at that point. Alternatively, we can see that the portion from $t = 60\,\mathrm{s}$ to $t = 120\,\mathrm{s}$ is relatively linear, so we can use that for our approximation, giving:

$$a\,(t = 90\,\mathrm{s}) \approx \frac{30 - 10}{120 - 60} = 0.33\,\mathrm{m/s}^2$$

Example 1.4 ($F = ma$ 2008)**.** The graph below shows the velocity-time graph of a car moving in a straight line. What is the maximum displacement from the start for the car?

Solution. The maximum displacement is the shaded area under the velocity-time graph below:



This is the maximum area under the graph we can include; if we further include any of the area from $t = 2.5\,\mathrm{s}$ to $t = 3\,\mathrm{s}$, the displacement will start decreasing, since the velocity is negative there (and the car moves backwards).

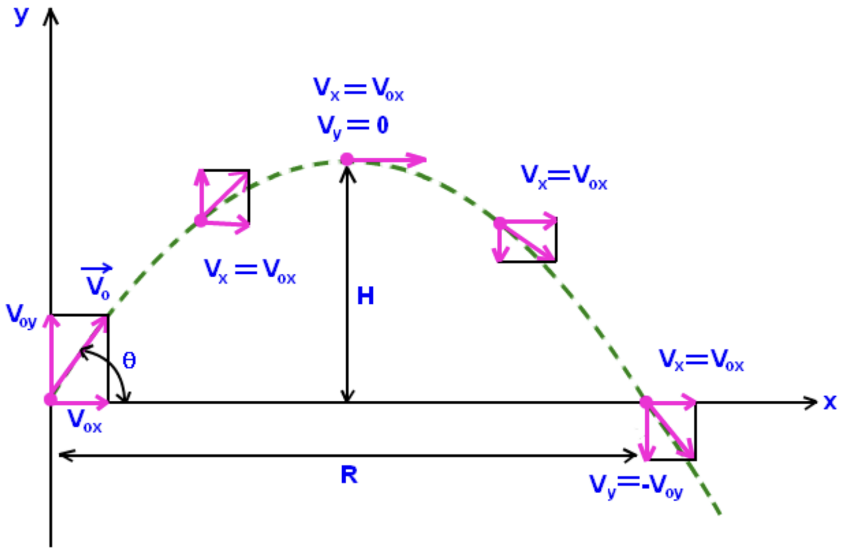
By some simple geometry, the area (and hence the maximum displacement) is:

$$x_{\max} = \frac{1}{2}\,(2+4)\,(1) + (4)\,(0.5) + \frac{1}{2}\,(4)\,(1) = 7\,\mathrm{m}$$

1.2 2D Kinematics: Projectile Motion

Now, we shall upgrade to 2D, where motion is now constrained to be along a plane. We first study the case of projectiles.

A projectile is an object in the air (neglecting drag) with two components of velocity, namely the x and y-components. We usually call the x-component the horizontal direction and the y-component the vertical direction. We also assume the projectile starts at the origin $(0, 0)$.



The key idea that we will use in our analysis is that we can analyse the motion in each direction **independently!** That is, motion in the x-direction will not affect the y-direction, and vice-versa. Hence, every projectile motion question is essentially two 1D problems.

1.2.1 Kinematic Equations For Each Direction

We know that the only acceleration of the particle is due to the gravitational acceleration, $\boldsymbol{g}$, which acts vertically downwards. Taking upwards and rightwards as positive, the relevant accelerations are $a_x = 0$ and $a_y = -g$.

Using Equation (8) to describe the motion in each direction, we have:

$$x = v_{x,\mathrm{i}} t = (v \cos \theta)\, t \tag{11}$$

$$y = v_{y,\mathrm{i}} t - \frac{1}{2} g t^2 = (v \sin \theta)\, t - \frac{1}{2} g t^2 \tag{12}$$

where $v_{x,i}$ and $v_{y,i}$ are the x and y-components of the initial velocity, and with some simple trigonometry, if v is the magnitude of the initial velocity and θ is its angle above the horizontal, then $v_{x,\mathrm{i}} = v \cos \theta$ and $v_{y,\mathrm{i}} = v \sin \theta$.

Now, we can also ask: what shape is the trajectory? To do this, we need to find $y(x)$, i.e. y as a function of x. Looking at Equations (11) and (12), the best way to do this is to get rid of t, which is easiest done by substituting $t = \frac{x}{v \cos \theta}$ from Equation (11) into Equation (12).

Upon doing so (and a bit of algebraic cleaning up), we obtain:

$$y = x \tan \theta - \frac{g x^2}{2 v^2 \cos^2 \theta} \tag{13}$$

which takes on the form of a **parabola** (as you might have guessed).

1.2.2 Parameters Of Projectile Motion

There are a few parameters we are concerned with when we talk about projectile motion on a flat ground. These are:

1. **Range:** Usually denoted by R, this is the horizontal distance travelled by the projectile before it hits the ground.
2. **Maximum Height:** Usually denoted by H, this is the height of the highest point the projectile reaches in its motion.
3. **Time Of Flight:** Usually denoted by T, this is the time taken between when the projectile is shot and when the projectile lands on the ground.

The range can be found by setting $y = 0$ (since that is the point when the projectile reaches the ground). From Equation (13), we have:

$$x \tan\theta - \frac{gx^2}{2v^2\cos^2\theta} = 0 \quad \Longrightarrow \quad x = 0\,(\text{reject})\ \text{or}\ x = \frac{2v^2\sin\theta\cos\theta}{g} \tag{14}$$

We reject $x = 0$ as this corresponds to the initial point where the projectile is first launched. Hence, the range is

$$R = \frac{2v^2\sin\theta\cos\theta}{g} = \frac{v^2\sin(2\theta)}{g} \tag{15}$$

where we can make use of the identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

By symmetry of the parabola, the maximum height occurs at $x = \frac{R}{2} = \frac{v^2\sin\theta\cos\theta}{g}$. Substituting this value of x into Equation (13), we obtain:

$$H = \left(\frac{v^2\sin\theta\cos\theta}{g}\right)(\tan\theta) - \frac{g}{2v^2\cos^2\theta}\left(\frac{v^2\sin\theta\cos\theta}{g}\right)^2 = \frac{v^2\sin^2\theta}{2g} \tag{16}$$

Lastly, we can revert to Equation (11) to find the time of flight, which occurs when $x = R$:

$$T = \frac{R}{v\cos\theta} = \frac{1}{v\cos\theta}\left(\frac{2v^2\sin\theta\cos\theta}{g}\right) = \frac{2v\sin\theta}{g} \tag{17}$$

Remark. The expressions in Equations (15) to (17) are only true **if the projectile motion takes place entirely on the same flat ground!** That is, the projectile must start and end at the same height. If this is not the case, you need to begin from first principles, i.e. Equations (11) to (13).

Example 1.5. A projectile is launched on a flat ground with initial speed 30 m/s. Given that the angle of launch can be freely adjusted, what is the maximum possible range of the projectile?

Solution. This is a simple application of Equation (15). The maximum value of R is achieved when the maximum value of $\sin(2\theta)$ is achieved. This happens when $\sin(2\theta) = 1$, which is achieved when $\theta = 45°$. Hence,

$$R_{\text{max}} = \frac{v^2}{g} = \frac{30^2}{9.81} = 92\,\text{m}$$

Example 1.6 (SJPO 2009, modified)**.** Two projectiles are launched with identical speeds of 30 m/s at angles of 40° and 50° with the horizontal, respectively. (a) Find the difference in their times of flight. (b) Find the difference in their ranges.

Solution. (a) This is a simple application of Equation (17). The difference is

$$\Delta T = \frac{2v}{g}(\sin\theta_1 - \sin\theta_2) = \frac{2\,(30)}{9.81}(\sin 50^\circ - \sin 40^\circ) = 0.75\,\text{s}$$

(b) This is a simple application of Equation (15). The difference is

$$\Delta R = \frac{v^2}{g}(\sin(2\theta_1) - \sin(2\theta_2)) = \frac{30^2}{9.81}(\sin 100^\circ - \sin 80^\circ) = 0\,\text{m}$$

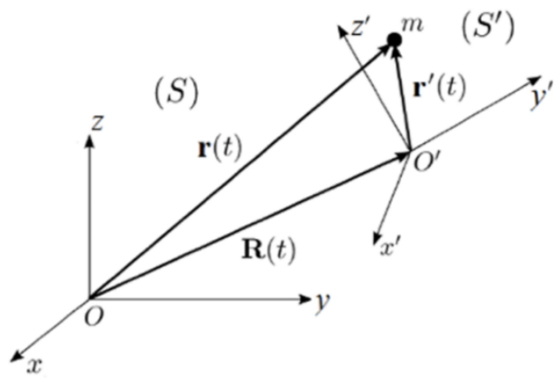
In general, if two projectiles on flat ground have the same launch speed and have complementary launch angles (i.e. their launch angles add up to 90°), they will have the same range!

1.3 Relative Motion

A **reference frame** is a coordinate system used by an observer, to describe the position, velocity and acceleration of objects relative to himself.

For instance, suppose that you are in a car moving at 30 m/s relative to the ground. You throw a ball at a speed of 20 m/s relative to yourself. Logically, you would expect the ground (or someone outside the car who is stationary) to see the ball moving at $20 + 30 = 50$ m/s. This is the idea behind **relative motion.**

To formalise this, consider the two frames of reference S and S' below, adopting the coordinate systems (x, y, z) and (x', y', z') respectively.



The observer in S measures a particle to be located at $\boldsymbol{r}$ in his frame, while the observer in S' measures the same particle to be located at $\boldsymbol{r'}$ in his frame. Note that these two vectors are different, because the origins of S and S' are separated by some vector $\boldsymbol{R}$. Then, by a simple vector addition, we have:

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r'} \tag{18}$$

We can make a similar claim regarding the velocities. Suppose that S' is moving at a velocity of $\boldsymbol{V}$ relative to S, and S and S' measure the velocity of the same particle to be $\boldsymbol{v}$ and $\boldsymbol{v'}$ respectively. Then, we have:

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{v'} \tag{19}$$

And, you can expect another similar claim regarding the accelerations:

$$\boldsymbol{a} = \boldsymbol{A} + \boldsymbol{a'} \tag{20}$$

Remark. When many things are moving at once, it may be useful to go into a reference frame whereby some objects become **static,** or one whereby motion becomes more **symmetric.**

Example 1.7 (SJPO 2013, modified)**.** A person rides on a bicycle at 10 m/s heading eastwards. He feels that the wind is blowing from the north at 10 m/s. Find the velocity (magnitude and direction) of the wind relative to the ground.

Solution. The two relevant frames of reference in this problem are that of the person (on the bicycle) and that of the ground. Let's take eastwards and northwards to be the positive x and y-directions respectively. Then, using the notation in Equation (19), we have $\boldsymbol{V} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$ m/s and $\boldsymbol{v'} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$ m/s. Hence,

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{v'} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \text{ m/s}$$

This means that the velocity of the wind relative to the ground has a magnitude of $\sqrt{10^2 + 10^2} = 10\sqrt{2} = 14.1$ m/s, at a direction of $\tan^{-1}\left(\frac{10}{10}\right) = 45°$ (measured anti-clockwise from the positive x-direction).

1.4 Dynamics Part 1

Now, we shall study **dynamics,** which concerns forces (the origin of motion). All of dynamics boils down to understanding Newton's Laws of Motion.

Newton's 1st Law: If an object has constant velocity (both magnitude and direction), the net force on it must be zero.

This one is rather self-explanatory. The reverse of the statement is also true. Do take note that it is possible for there to be external forces on the object, so long as the sum of these forces cancels out.

Newton's 2nd Law: Often called N2L. If the mass of an object is constant, then the net force on it is equal to the product of its mass and its acceleration:

$$\boldsymbol{F}_{\text{net}} = m\boldsymbol{a} \tag{21}$$

This implies that $\boldsymbol{F}_{\text{net}}$ is parallel to $\boldsymbol{a}$.

Newton's 3rd Law: Every action has an equal and opposite reaction force, forming an action-reaction pair.

A valid action-reaction pair of forces must also satisfy the following:

1. Acts on different bodies (A on B, and B on A)
2. Has the same nature

Example 1.8. A book lies on a table. Is the weight of the book and the normal force exerted by the table on the book an action-reaction pair?

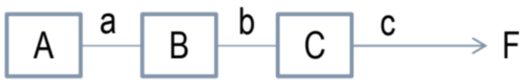
Solution. No, they are not of the same nature. The two action-reaction pairs here are:

1. The gravitational force exerted by the Earth on the book and the gravitational force exerted by the book on the Earth.
2. The normal force exerted by the table on the book and the normal force exerted by the book on the table.

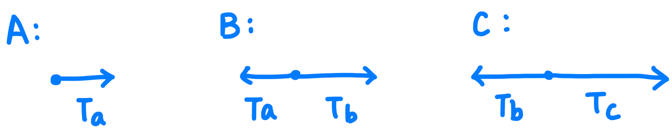
A very powerful tool in dynamics to help us keep track of forces is the **free-body diagram** (or FBD). In a FBD, we draw out all the forces **acting on** the object.

1.4.1 Using N2L

Example 1.9. Three boxes A (4 kg), B (2 kg) and C (1 kg) are connected by light, inextensible strings and are pulled by a force F on a frictionless floor. If the acceleration of box C is $2\,\mathrm{m/s^2}$, what is the tension in string a, b and c respectively?



Solution. It will help to start by drawing the FBDs of each mass. We will ignore the gravitational and normal forces (we know that these forces will cancel anyway).



Since the three boxes must move as a whole, the accelerations of all three boxes must be the same at $2\,\mathrm{m/s^2}$. Hence, writing N2L for each block,

$$T_\mathrm{c} - T_\mathrm{b} = m_\mathrm{C} a = 2\,\mathrm{N}$$

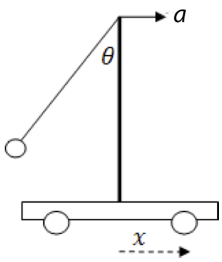
$$T_\mathrm{b} - T_\mathrm{a} = m_\mathrm{B} a = 4\,\mathrm{N}$$

$$T_\mathrm{a} = m_\mathrm{A} a = 8\,\mathrm{N}$$

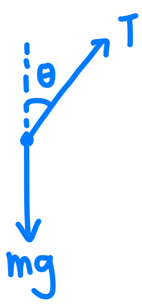
Solving these equations gives the desired tensions of $T_\mathrm{a} = 8\,\mathrm{N}$, $T_\mathrm{b} = 12\,\mathrm{N}$, $T_\mathrm{c} = 14\,\mathrm{N}$.

Now, let's look at a harder 2D example.

Example 1.10 (SJPO 2009)**.** A small bob with a mass of 250 g is suspended by a string from a clamp attached to a cart that is accelerating at a constant rate of $2\,\mathrm{m/s^2}$ at it moves along a flat straight table as shown in the diagram below. Assuming the bob to be suspended motionless with respect to the cart, what angle does the string make with the vertical?



Solution. Again, it will help to start by drawing the FBD of the bob.



Writing $F_{\text{net}} = ma$ in the horizontal direction, we have:

$$T \sin\theta = ma$$

The bob is in equilibrium in the vertical direction, hence we have:

$$T \cos\theta = mg$$

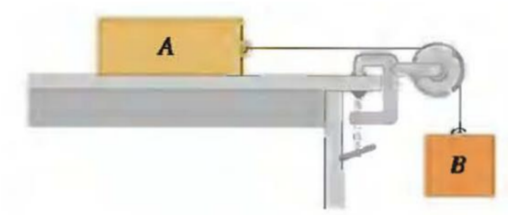
We can eliminate T by dividing the first equation by the second:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{ma}{mg} \implies \tan\theta = \frac{a}{g} \implies \theta = \tan^{-1}\left(\frac{a}{g}\right) = \tan^{-1}\left(\frac{2}{9.81}\right) = 11.5^\circ$$

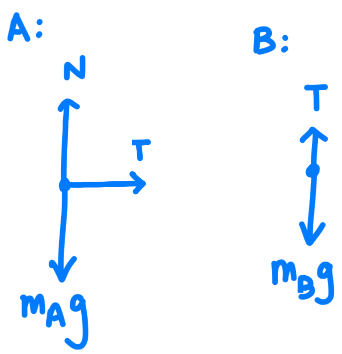
1.4.2 Pulley Problems

Pulley problems can be solved by systematically drawing out all the FBDs, being careful with tensions, and solving a system of simultaneous equations.

Example 1.11. In the set-up below, $m_{\text{A}} = 20\,\text{kg}$, $m_{\text{B}} = 10\,\text{kg}$, and the pulley and set-up are frictionless. Suppose the string is light and inextensible. (a) Find the acceleration of the system of masses. (b) Find the tension in the string. (*Note: A light string means that the string is massless, while an inextensible string means that the length of the string remains constant while it is in tension, and hence the tension is the same through the string.*)



Solution. (a) We first draw the FBDs of both masses:



Writing the N2L equations,

$$\text{Vertical, Block B:} \quad m_{\text{B}}g - T = m_{\text{B}}a$$

$$\text{Horizontal, Block A:} \quad T = m_{\text{A}}a$$

(The vertical N2L equation for Block A doesn't matter for this question.)

Adding the two equations, we obtain:

$$m_{\text{B}}g = (m_{\text{A}} + m_{\text{B}})\,a \implies a = \frac{m_{\text{B}}g}{m_{\text{A}} + m_{\text{B}}} = \frac{(10)\,(9.81)}{20+10} = 3.27\,\text{m/s}^2$$

Equivalently, you can also treat Block A and B combined as one system with total mass $m_\text{A}+m_\text{B}$. The only external force acting on this combined system that is causing it to accelerate is Block B's weight of $m_\text{B}g$, which explains the result for the acceleration above.

(b) The tension in the string can be found using either N2L equation:

$$T=\frac{m_\text{A}m_\text{B}g}{m_\text{A}+m_\text{B}}=\frac{(20)\,(10)\,(9.81)}{20+10}=65.4\,\text{N}$$

1.4.3 Apparent Weight

When you stand on a weighing scale, the reading on the weighing scale is *not* actually your weight! It is the **normal force** between you and the scale.

Of course, when you and the scale are in equilibrium, this normal force is equivalent to your weight. However, under a situation whereby you and the scale are accelerating, the normal force will no longer be the same as your weight.

Example 1.12. You (of mass m) are standing on a weighing scale in an elevator that is accelerating with an acceleration of a. (a) If the elevator is accelerating upwards, find the reading on the weighing scale. (b) Do the same for the case of the elevator accelerating downwards. (c) What should be the acceleration of the elevator for you to feel "weightless"?

Solution. (a) Let the normal force (scale reading) be N. The net force is ma upwards. Hence,

$$N-mg=ma \qquad\Longrightarrow\qquad N=m\,(g+a)$$

(b) The net force is ma downwards. Hence,

$$mg-N=ma \qquad\Longrightarrow\qquad N=m\,(g-a)$$

(c) You will feel "weightless" if $N=0$ (i.e. you have 0 apparent weight). This can only be achieved for the case where the elevator is accelerating downwards, and it happens when $a=g$.

1.4.4 Friction

Friction exists when two surfaces are in contact with each other, and it opposes the slipping or tendency to slip between the two surfaces.

There are two main types of friction we consider:

1. **Static friction:** Exists between two surfaces which have **yet to start slipping.** It prevents them from starting to slip.
2. **Kinetic friction:** Exists between two surfaces which have **already started slipping.** It opposes their slipping motion.

Two important parameters are the **coefficient of static friction** μ_s and the **coefficient of kinetic friction** μ_k. These depend on the properties of the surfaces (and will be given to you).

The two frictional forces are given by:

$$f_\text{s}\le\mu_\text{s}N \tag{22}$$

$$f_\text{k}=\mu_\text{k}N \tag{23}$$

where N is the normal force between the two surfaces.

Equation (23) is rather self-explanatory: the kinetic friction is constant. However, the **inequality sign** in Equation (22) is very important! It indicates that static friction can range from 0 to the maximum value of $\mu_\text{s}N$. This is because the magnitude of static friction must be exactly large enough to oppose the external forces acting on the object, so that the object stays in equilibrium. If the external force is less than $\mu_s N$, then the static friction must be also less than it.

Two surfaces will just begin to slip when the equality in Equation (22) is met:

$$f_{s,slip} = \mu_s N \tag{24}$$

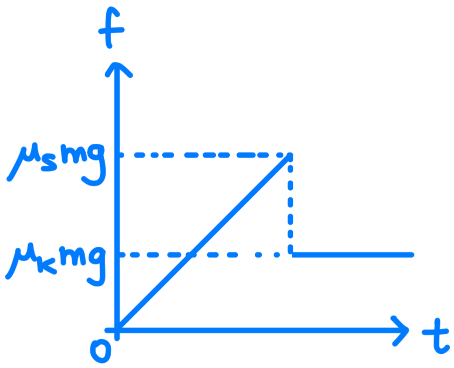
Remark. Usually, we have $0 < \mu_k < \mu_s$. However, in some problems, there is no distinction made between μ_s and μ_k, or it could be simply be called the coefficient of friction μ. In these cases, we shall assume $\mu_s = \mu_k = \mu$. This is usually an assumption made to simplify problems.

Example 1.13. Consider a block of mass m sitting on a rough horizontal surface. A horizontal external force F is applied to the block, and the force F is increasing linearly with time (starting from $F = 0$ when $t = 0$). Make a sketch of the frictional force on the block against the time elapsed. Assume that $0 < \mu_k < \mu_s$.

Solution. Initially, the friction is static (and the block doesn't move). The static friction must be just enough to balance the external force on the block. Since the external force is linearly increasing, the static friction must be linearly increasing too.

This happens until the critical value of the static friction as per Equation (24), $f_{s,slip} = \mu_s N = \mu_s mg$. At this point, the block starts slipping with respect to the surface, and friction becomes kinetic. The kinetic friction is constant and is given by $f_k = \mu_k N = \mu_k mg$.

With the above information, the required graph is (note the discontinuity):



Example 1.14. Solve Example 1.11 again, but this time, assume that there is a coefficient of kinetic friction $\mu_k = 0.20$ between Block A and the table.

Solution. We can adopt a similar method, except the N2L equation for Block A becomes:

$$T - f_k = m_A a \quad \Longrightarrow \quad T - \mu_k m_A g = m_A a$$

Adding this with the vertical N2L equation for Block B (which remains the same), we get

$$m_B g - \mu_k m_A g = (m_A + m_B)\, a$$

$$a = \frac{m_B g - \mu_k m_A g}{m_A + m_B} = \frac{(10)\,(9.81) - (0.20)\,(20)\,(9.81)}{20 + 10} = 1.96\,\text{m/s}^2$$

$$T = m_B g - m_B a = \frac{(\mu_k + 1)\, m_A m_B g}{m_A + m_B} = 78.5\,\text{N}$$

You can do a simple sanity check of your results by substituting $\mu_k = 0$, and verifying that the expressions are the same as those found in Example 1.11.

1.4.5 Drag & Fluid Resistance

There are two main models of fluid resistance which we shall consider.

For **lower speeds,** the drag force is **linear** in the velocity:

$$f = kv \tag{25}$$

For **higher speeds,** the drag force is **quadratic** in the velocity:

$$f = Dv^2 \tag{26}$$

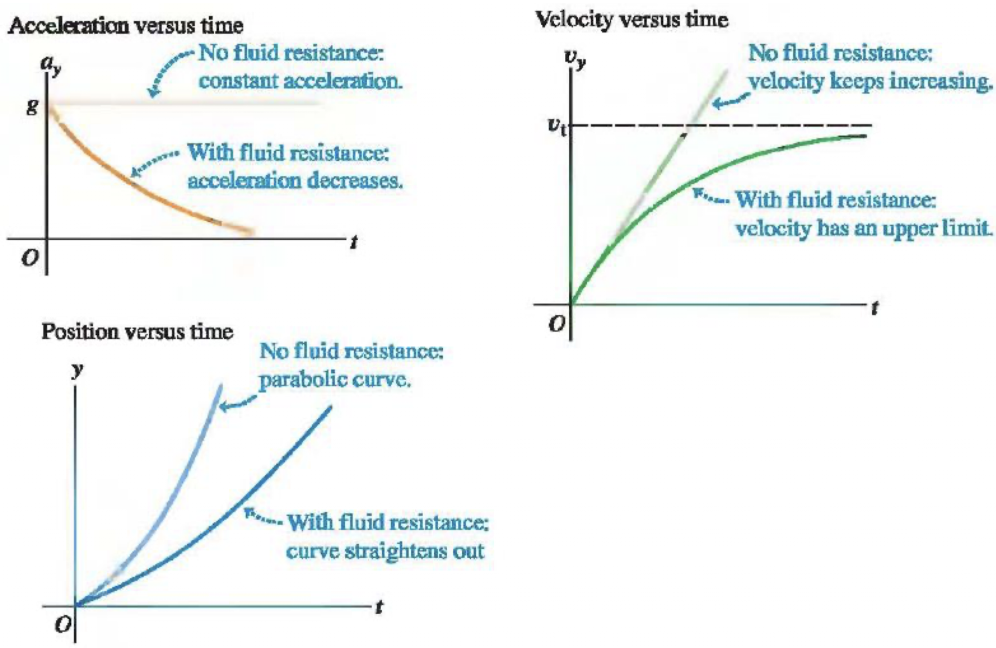
where k and D are constants.

More often than not, we are interested in finding the **terminal velocity** v_{T} of an object. This is the constant velocity at which the object experiences no net force.

For an object of mass m falling vertically, the terminal velocities are:

$$mg = kv_{\mathrm{T}} \quad \Longrightarrow \quad v_{\mathrm{T}} = \frac{mg}{k} \tag{27}$$

$$mg = D{v_{\mathrm{T}}}^2 \quad \Longrightarrow \quad v_{\mathrm{T}} = \sqrt{\frac{mg}{D}} \tag{28}$$



1.4.6 Spring Force & Spring Combinations

Springs usually start with some **natural length** x_0. This is the length of the spring when no net external force acts on it. However, in some cases, we may simplify problems by setting $x_0 = 0$.

We shall restrict our discussion to ideal springs, which obey **Hooke's Law:**

$$\boldsymbol{F}_{\text{spring}} = -k\boldsymbol{x} \tag{29}$$

Here, $\boldsymbol{x}$ is the **compression/extension** of the spring, defined as the difference in length between the current length and the natural length, and k is the spring constant.

Remark. The negative sign in Equation (29) is essential, because it signifies that the direction of the spring force is opposite to the direction of compression/extension, meaning that the spring always wants to restore itself back to its original length!

Example 1.15. Consider a spring of natural length $x_0 = 5\,\text{cm}$ and spring constant $k = 1.5\,\text{N/cm}$. (a) What is the force required to stretch the spring to a length of 7 cm? (b) How about 9 cm? (c) What is the additional force required to stretch the spring from 7 cm to 9 cm?

Solution. (a) This just relies on repeated simple applications of Equation (29). To stretch from 5 cm to 7 cm, we need a force of

$$F_{5\text{ to }7} = kx_{5\text{ to }7} = (1.5)\,(7-5) = 3\,\text{N}$$

(b) To stretch from 5 cm to 9 cm, we need a force of

$$F_{5\text{ to }9} = kx_{5\text{ to }9} = (1.5)\,(9-5) = 6\,\text{N}$$

(c) This is just the difference between the two required forces in (a) and (b), which is

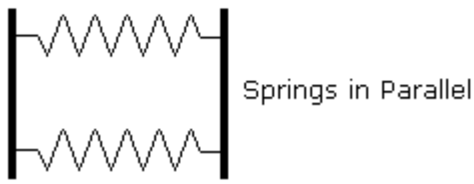
$$F_{7\text{ to }9} = F_{5\text{ to }9} - F_{5\text{ to }7} = 6 - 3 = 3\,\text{N}$$

However, this also happens to be the same if we just did $(1.5)\,(9-7) = 3\,\text{N}$! This shows that due to the linearity of the spring, we can "bypass" the natural length for calculations like these.

In more complicated scenarios, you may see springs being combined together. The trick for these kinds of questions is to always replace complicated arrangements of springs with just one spring of some **effective spring constant** k_{eff}.

For springs in **parallel** (i.e. joined end-to-end), the effective spring constant is:

$$k_{\text{eff}} = \sum_i k_i \tag{30}$$



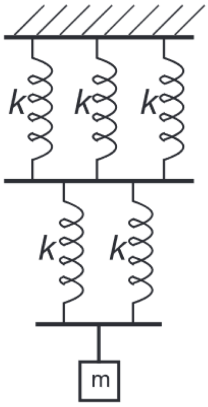
This is just a simple, algebraic sum.

For springs in **series** (i.e. joined side-by-side), the effective spring constant is:

$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i} \tag{31}$$

This case is not as simple. First, we take the reciprocals of each spring constant. Then, we sum up the reciprocals. Lastly, the effective spring constant is the reciprocal of the sum of reciprocals!

Example 1.16. Find the effective spring constant of the following set-up. All springs are identical and have spring constant k each.

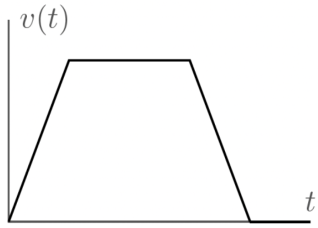


Solution. The top three springs are in parallel with one another, forming an effective spring constant of $3k$. The bottom two springs are in parallel with one another, forming an effective spring constant of $2k$. Collectively, the top three springs are in series with the bottom two springs. Hence, the effective spring constant is

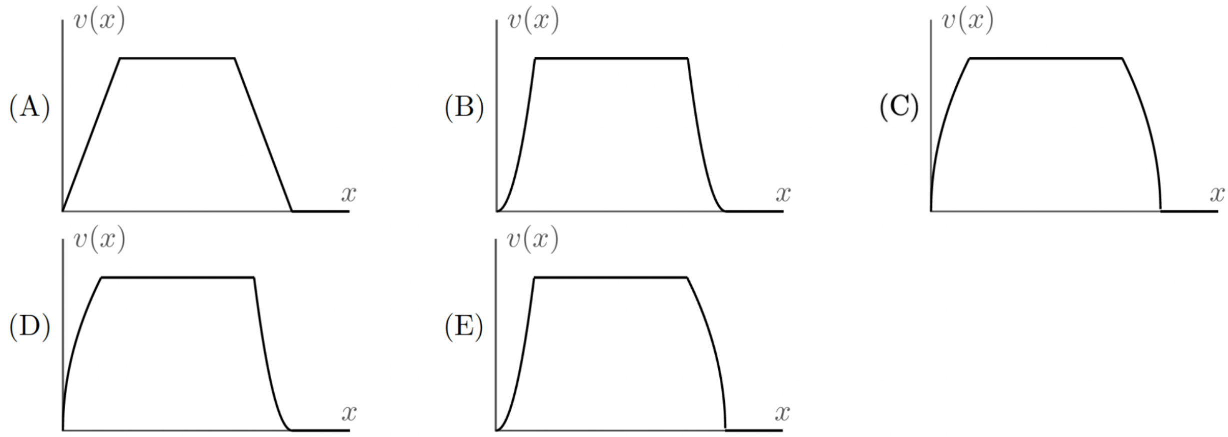
$$k_{\text{eff}} = \frac{1}{\frac{1}{3k} + \frac{1}{2k}} = \frac{6}{5}k$$

2 Problems

Problem 2.1 ($F = ma$ 2024)**.** A particle travels in a straight line. Its velocity-time graph is shown below.



Which of the following shows the graph of the velocity as a function of displacement from its initial position?



Solution. (C)

Problem 2.2 ($F = ma$ 2020)**.** A ball is bouncing vertically between a floor and ceiling, which are both horizontal and separated by 4 m. All collisions are perfectly elastic, and when the ball hits the floor, it has a speed of 12 m/s. How long does a complete up-down cycle take?

(A) 0.3 s

(B) 0.4 s

(C) 0.6 s

(D) 0.8 s

(E) 2.4 s

Solution. (D)

Problem 2.3 (SJPO 2011)**.** A car accelerates from 4.0 m/s to 24 m/s at a rate of 3.0 m/s^2. How far does it travel while accelerating?

(A) 56 m

(B) 93 m

(C) 158 m

(D) 195 m

(E) 279 m

Solution. (B)

Problem 2.4 (SJPO 2011)**.** A marathon runner runs at a steady speed of 15 km/h. When the runner is 7.5 km from the finish, a bird begins flying from the runner to the finish at 30 km/h. When the bird reaches the finish line, it turns around and flies back to the runner, and then turns around again, repeating the back-and-forth trips until the runner reaches the finish line. How many kilometres does the bird travel?

(A) 10 km

(B) 15 km

(C) 20 km

(D) 30 km

(E) 45 km

Solution. (B)

Problem 2.5 (SJPO 2010)**.** Consider a train that can speed up with an acceleration of $20\,\text{cm/s}^2$ and slow down with a deceleration of $100\,\text{cm/s}^2$. Find the minimum time for the train to travel between two stations 2 km apart. You may assume that the train has to stop at every station.

(A) 33.3 s

(B) 57.7 s

(C) 81.6 s

(D) 141 s

(E) 155 s

Solution. (E)

Problem 2.6 (SJPO 2010)**.** Two identical balls are at rest and side by side at the top of a hill. You let one ball, A, start rolling down the hill. A little later, you start the second ball, B down the hill by giving it a shove. The second ball rolls down the hill along a line parallel to the path of the first ball and passes it. At the instant ball B passes ball A,

(A) only the displacement and velocity are the same for both balls.

(B) only the displacement and acceleration are the same for both balls.

(C) only the velocity and acceleration are the same for both balls.

(D) the displacement, velocity and acceleration are the same for both balls.

(E) the displacement, velocity and acceleration are all different for both balls.

Solution. (B)

Problem 2.7 (SJPO 2012)**.** An object falls freely through the air. What is the ratio of the distance fallen in the 1st second to the distance fallen in the 2nd second to the distance fallen in the 3rd second? You may neglect air resistance.

(A) 1 : 2 : 3

(B) 1 : 4 : 9

(C) 1 : 3 : 5

(D) $1 : \sqrt{2} : \sqrt{3}$

(E) $1 : \sqrt{2} - 1 : \sqrt{3} - 1$

Solution. (C)

Problem 2.8 (SJPO 2013)**.** A boat is travelling in a river at speed v_1 relative to still water, with the speed of the water current as v_2. When the boat steers perpendicularly to the bank at speed v_1, it reaches the opposite bank in time t_1. When the boat steers at an angle upstream such that it will reach directly opposite of its starting point with the same speed, the time taken is t_2. What is the ratio of v_1 to v_2?

(A) $\frac{t_1}{\sqrt{{t_1}^2+{t_2}^2}}$

(B) $\frac{t_2}{\sqrt{{t_1}^2+{t_2}^2}}$

(C) $\frac{t_1}{\sqrt{{t_2}^2-{t_1}^2}}$

(D) $\frac{t_2}{\sqrt{{t_2}^2-{t_1}^2}}$

(E) None of the above

Solution. (D)

Problem 2.9 (SJPO 2014)**.** Before a train enters a tunnel, it is important that it sounds its horn. The train velocity is 80 km/h and the speed of sound is taken to be 340 m/s. The echo of the horn signal is reflected off the surface of the cliff through which the tunnel runs. The echo is heard 2.0 s after the driver sounds the horn. The distance of the train from the cliff at the time when the echo is heard is

(A) 318 m

(B) 260 m

(C) 362 m

(D) 159 m

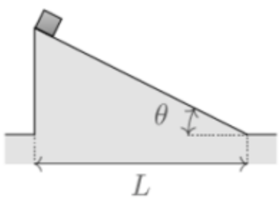
(E) 636 m

Solution. (A)

Problem 2.10. A ball is kicked at an angle of 35° above the ground so that it hits a target that is 30 m away at a height of 1.8 m. (a) What is the initial speed of the ball? (b) What is the time taken for the ball to reach the target? (c) What is the speed at which the ball hits the target?

Solution. (a) 18.7 m/s (b) 1.96 s (c) 17.7 m/s, using $g = 10\,\text{m/s}^2$

Problem 2.11 ($F = ma$ 2021)**.** A block is released from rest at the top a fixed, frictionless ramp with horizontal length L and inclination θ.



For a fixed value of L, which value of θ minimises the time needed for the block to reach the bottom of the ramp?

(A) 30°

(B) 45°

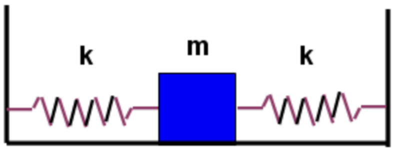
(C) 60°

(D) 75°

(E) 80°

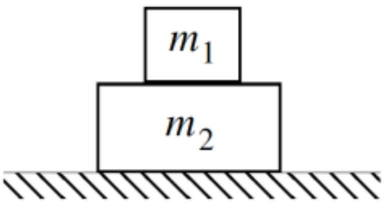
Solution. (B)

Problem 2.12. Two identical springs, each of spring constant k, are connected to each side of a mass m as shown below. Find an expression for the effective spring constant of the system. *This is probably not as simple as you think!*



Solution. $k_{\text{eff}} = 2k$. *Think: Why is it not* $\frac{1}{2}k$ *as per springs in series?*

Problem 2.13 ($F = ma$ 2018)**.** Two blocks of masses $m_1 = 2.0\,\text{kg}$ and $m_2 = 1.0\,\text{kg}$ are stacked together on top of a frictionless table as shown. The coefficient of static friction between the blocks is $\mu_s = 0.20$. What is the minimum horizontal force that must be applied to the top block to make it slide across the bottom block?



(A) 4.0 N

(B) 6.0 N

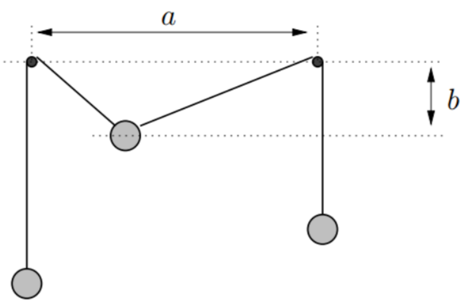
(C) 8.0 N

(D) 12.0 N

(E) The top block will never slide across the bottom block.

Solution. (D)

Problem 2.14 ($F = ma$ 2010)**.** The three masses as shown in the accompanying diagram are equal. The pulleys are small, the string is lightweight, and friction is negligible. Assuming the system is in equilibrium, what is the ratio $\frac{a}{b}$? The figure is not drawn to scale!



(A) $\frac{1}{2}$

(B) 1

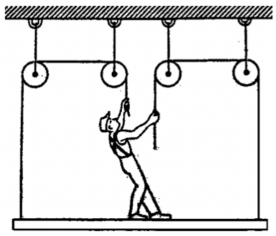
(C) $\sqrt{3}$

(D) 2

(E) $2\sqrt{3}$

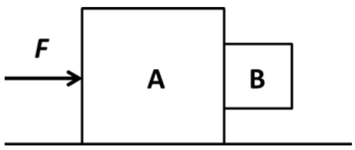
Solution. (E)

Problem 2.15. A painter of mass M stands on a platform of mass m and pulls himself up by two ropes which hang over pulleys, as shown in the figure below. He pulls each rope with force F and accelerates upward with a uniform acceleration a. Find a. *Be careful in counting the number of tensions!*



Solution. $a = \frac{4F}{M+m} - g$

Problem 2.16 (SJPO 2012)**.** For the system consisting of the 2 blocks shown in the figure below, a horizontal force F is applied so that block B does not fall due to its weight. The masses of A and B are 12.0 kg and 2.0 kg respectively. The horizontal surface is frictionless and the coefficient of static friction between the 2 blocks is 0.45. What is the minimum magnitude of F?



(A) $\frac{63}{10}g$

(B) $\frac{100}{3}g$

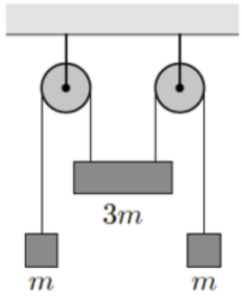
(C) $\frac{280}{9}g$

(D) $\frac{560}{9}g$

(E) $70g$

Solution. (C)

Problem 2.17 ($F = ma$ 2023)**.** Two blocks of mass m and a block of mass $3m$ are attached to a system of massless fixed pulleys and massless strings, as shown.



Assume all surfaces are frictionless. What is the acceleration of each mass m?

(A) $\frac{g}{8}$

(B) $\frac{g}{5}$

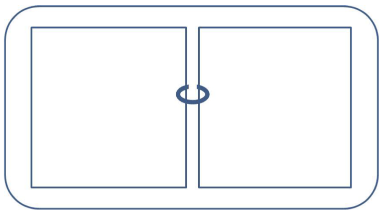
(C) $\frac{g}{4}$

(D) $\frac{g}{3}$

(E) $\frac{2g}{3}$

Solution. (B)

Problem 2.18 (SJPO 2014)**.** As shown in the diagram, the setup comprises of a metal box with a metal pole at the centre. The mass of the metal box and pole is M. A ring of mass m slides down the pole with an acceleration a. The frictional force between the ring and the pole is f. What is the force of the box on the floor, when the ring is sliding down?



(A) Mg

(B) $(M + m)\,g$

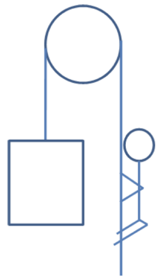
(C) $Mg + f$

(D) $(M + m)\,g - f$

(E) $(M + m)\,a$

Solution. (C)

Problem 2.19 (SJPO 2013)**.** In the diagram below, the man (of mass m) is at the same initial horizontal level as the counterweight (also of mass m). As the man accelerates and starts to climb up the rope, which of the following correctly reflects the motion of the man and the counterweight? Ignore the mass of the rope and the friction in the pulley system.



(A) The man accelerates upwards and the counterweight accelerates downwards.

(B) The man accelerates upwards and the counterweight does not move.

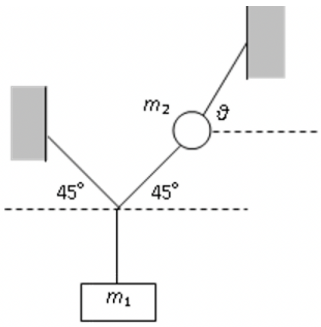
(C) The man and the weight accelerate upwards and reaches the ceiling at the same time.

(D) The man and the weight accelerate upwards but the man moves faster then the weight.

(E) The man and weight accelerate upwards but the weight moves faster than the man.

Solution. (C)

Problem 2.20. The set-up below is in equilibrium. Determine $\tan\theta$ in terms of m_1 and m_2.



Solution. $\tan\theta = 1 + \frac{2m_2}{m_1}$